Cyclic inequality with quadratic roots.

https://www.linkedin.com/feed/update/urn:li:activity:6709361692561719296

Let $x,y,z > 0$, prove that

$$\sqrt{\frac{x}{x+y}} + \sqrt{\frac{y}{y+z}} + \sqrt{\frac{z}{z+x}} \le \frac{3}{\sqrt{2}}.$$

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By Cauchy Inequality $\sum\sqrt{\frac{x}{x+y}} = \sum\sqrt{\frac{x}{(x+y)(z+x)}\cdot(z+x)} \le$

$$\sqrt{\sum\frac{x}{(x+y)(z+x)}}\cdot\sqrt{\sum(z+x)} = \sqrt{\sum\frac{x}{(x+y)(z+x)}}\cdot\sqrt{2(x+y+z)}.$$

Thus, remains to prove inequality

$$\sqrt{\sum\frac{x}{(x+y)(z+x)}}\cdot\sqrt{2(x+y+z)} \le \frac{3}{\sqrt{2}} \Leftrightarrow (x+y+z)\sum\frac{x}{(x+y)(z+x)} \le \frac{9}{4} \Leftrightarrow$$

$$\frac{x+y+z}{(x+y)(y+z)(z+x)}\cdot\sum x(y+z) \le \frac{9}{4} \Leftrightarrow \frac{(x+y+z)\cdot 2(xy+yz+zx)}{(x+y)(y+z)(z+x)} \le \frac{9}{4} \Leftrightarrow$$

(**1**) $8(x+y+z)(xy+yz+zx) \le 9(x+y)(y+z)(z+x)$.

Since $(x+y)(y+z)(z+x) = (x+y+z)(xy+yz+zx) - xyz$ then (**1**)$\Leftrightarrow$

$8(x+y+z)(xy+yz+zx) \le 9(x+y+z)(xy+yz+zx) - 9xyz \Leftrightarrow$

$9xyz \le (x+y+z)(xy+yz+zx)$, where latter holds because by AM-GM Inequality we have $x+y+z \ge 3(xyz)^{1/3}$ and $xy+yz+zx \ge 3(xyz)^{2/3}$.